

Affine combinations in affine schemes

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Introduction

The notion of “neighbour points” in algebraic geometry is a geometric rendering of the notion of nilpotent elements in commutative rings, and was developed since the time of Study, Kähler, Hjelmslev, and notably in French algebraic geometry (Grothendieck, Weil et al.) since the 1950s. They introduced it via what they call *the first neighbourhood of the diagonal*.

In [3], [4] and [8] the neighbour notion was considered on an axiomatic basis, essentially for finite dimensional manifolds; one of the aims was to describe a combinatorial theory of differential forms.

In the specific context of algebraic geometry, such theory of differential forms was developed in [1], where it applies not only to manifolds, but to arbitrary schemes.

One aspect, present in [4] and [8], but not in [1], is the possibility of forming affine combinations of finite sets of mutual (1st order) neighbour points. The present note completes this aspect, by giving the construction of such affine combinations, at least in the category of *affine* schemes (the dual of the category of finitely presented commutative rings or k -algebras).

The interest in having the possibility of such affine combinations is documented in several places in [8], and is in [4] the basis for constructing, for any manifold, a simplicial object, whose cochain complex is the deRham complex of the manifold.

From a more philosophical viewpoint, one may say that the possibility of having affine combinations, for sets of mutual neighbour points, expresses in a concrete way the idea that spaces are “infinitesimally like affine spaces”.

1 Neighbour maps between algebras

Let k be a commutative ring. Consider commutative k -algebras B and C and two k -algebra maps f and $g : B \rightarrow C$. We say that they are *neighbours*, or more completely, *(first order) infinitesimal neighbours*, if

$$(f(a) - g(a)) \cdot (f(b) - g(b)) = 0 \text{ for all } a, b \in B, \quad (1)$$

or equivalently, if

$$f(a) \cdot g(b) + g(a) \cdot f(b) = f(a \cdot b) + g(a \cdot b) \text{ for all } a, b \in B. \quad (2)$$

(Note that this latter formulation makes no use of “minus”.) When this holds, we write $f \sim g$ (or more completely, $f \sim_1 g$). The relation \sim is a reflexive and symmetric relation (but not transitive). If the element $2 \in k$ is invertible, a third equivalent formulation of $f \sim g$ goes

$$(f(a) - g(a))^2 = 0 \text{ for all } a \in B. \quad (3)$$

For, it is clear that (1) implies (3). Conversely, assume (3), and let $a, b \in B$ be arbitrary, and apply (3) to the element $a + b$. Then by assumption, and using that f and g are algebra maps¹,

$$\begin{aligned} 0 &= (f(a + b) - g(a + b))^2 = [(f(a) - g(a)) + (f(b) - g(b))]^2 \\ &= (f(a) - g(a))^2 + (f(b) - g(b))^2 - 2(f(a) - g(a)) \cdot (f(b) - g(b)). \end{aligned}$$

The two first terms are 0 by assumption, hence so is the third. Now divide by 2.

Note that if C has no zero-divisors, then $f \sim g$ is equivalent to $f = g$.

It is clear that the relation \sim is stable under precomposition:

$$\text{if } h : B' \rightarrow B \text{ and } f \sim g : B \rightarrow C, \text{ then } f \circ h \sim g \circ h : B' \rightarrow C, \quad (4)$$

and, using that h is an algebra map, it is also stable under postcomposition:

$$\text{if } h : C \rightarrow C' \text{ and } f \sim g : B \rightarrow C, \text{ then } h \circ f \sim h \circ g : B \rightarrow C'. \quad (5)$$

¹“algebra” means throughout “commutative k -algebra”, and similarly for algebra maps. When we say “linear map”, we mean k -linear. By \otimes , we mean \otimes_k .

Also, if $h : B' \rightarrow B$ is a surjective algebra map, precomposition by h not only preserves the neighbour relation, it also reflects it, in the following sense

$$f \circ h \sim g \circ h \text{ implies } f \sim g. \quad (6)$$

This is immediate from (1); the a and b occurring there is of the form $h(a')$ and $h(b')$ for suitable a' and b' in B' , by surjectivity of h .

An alternative “element-free” formulation of the neighbour relation (Proposition 1.2 below) comes from a standard piece of commutative algebra. Recall that for commutative k -algebras A and B , the tensor product $A \otimes B$ carries structure of commutative k -algebra ($A \otimes B$ is in fact a coproduct of A and B); the multiplication map $m : B \otimes B \rightarrow B$ is a k -algebra homomorphism; so the kernel is an ideal $J \subseteq B \otimes B$.

The following is a classical description of the ideal $J \subseteq B \otimes B$; we include it for completeness.

Proposition 1.1 *The kernel J of $m : B \otimes B \rightarrow B$ is generated by the expressions $1 \otimes b - b \otimes 1$, for $b \in B$. Hence the ideal J^2 is generated by the expressions $(1 \otimes a - a \otimes 1) \cdot (1 \otimes b - b \otimes 1)$ (or equivalently, by the expressions $1 \otimes ab + ab \otimes 1 - a \otimes b - b \otimes a$).*

Proof. It is clear that $1 \otimes b - b \otimes 1$ is in J . Conversely, assume that $\sum_i a_i \otimes b_i$ is in J , i.e. that $\sum_i a_i \cdot b_i = 0$. Rewrite the i th term $a_i \otimes b_i$ as follows:

$$a_i \otimes b_i = a_i b_i \otimes 1 + (a_i \otimes 1) \cdot (1 \otimes b_i - b_i \otimes 1)$$

and sum over i ; since $\sum_i a_i b_i = 0$, we are left with $\sum_i (a_i \otimes 1) \cdot (1 \otimes b_i - b_i \otimes 1)$, which belongs to the $B \otimes B$ -module generated by elements of the form $1 \otimes b - b \otimes 1$. – The second assertion follows, since $ab \otimes 1 + 1 \otimes ab - a \otimes b - b \otimes a$ is the product of the two generators $1 \otimes a - a \otimes 1$ and $1 \otimes b - b \otimes 1$. (Note that the proof gave a slightly stronger result, namely that J is generated already as a B -module, by the elements $1 \otimes b - b \otimes 1$, via the algebra map $i_0 : B \rightarrow B \otimes B$, where $i_0(b) = b \otimes 1$).

From the second assertion in this Proposition immediately follows that $f \sim g$ iff $\{f, g\} : B \otimes B \rightarrow C$ factors across the quotient map $B \otimes B \rightarrow (B \otimes B)/J^2$ (where $\{f, g\} : B \otimes B \rightarrow C$ denotes the map given by $a \otimes b \mapsto f(a) \cdot g(b)$); equivalently:

Proposition 1.2 *For $f, g : B \rightarrow C$, we have $f \sim g$ if and only if $\{f, g\} : B \otimes B \rightarrow C$ annihilates J^2 .*

The two natural inclusion maps i_0 and $i_1 : B \rightarrow B \otimes B$ (given by $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$, respectively) are not in general neighbours, but when postcomposed with $\pi : B \otimes B \rightarrow (B \otimes B)/J^2$, they are:

$$\pi \circ i_0 \sim \pi \circ i_1,$$

and this is in fact the universal pair of neighbour algebra maps with domain B .

2 Neighbours for polynomial algebras

We consider the polynomial algebra $B := k[X_1, \dots, X_n]$. Identifying $B \otimes B$ with $k[Y_1, \dots, Y_n, Z_1, \dots, Z_n]$, the multiplication map m is the algebra map given by $Y_i \mapsto X_i$ and $Z_i \mapsto X_i$, so it is clear that the kernel J of m contains the n elements $Z_i - Y_i$. The following Proposition should be classical:

Proposition 2.1 *The ideal $J \subseteq B \otimes B$, for $B = k[X_1, \dots, X_n]$, is generated (as a $B \otimes B$ -module) by the n elements $Z_i - Y_i$.*

Proof. From Proposition 1.1, we know that J is generated by elements $P(\underline{Z}) - P(\underline{Y})$, for $P \in k[\underline{X}]$ (where \underline{X} denotes X_1, \dots, X_n , and similarly for \underline{Y} and \underline{Z}). So it suffices to prove that $P(\underline{Z}) - P(\underline{Y})$ is of the form

$$\sum_{i=1}^n (Z_i - Y_i) Q_i(\underline{Y}, \underline{Z}).$$

This is done by induction in n . For $n = 1$, it suffices, by linearity, to prove this fact for each monomial X^s . And this follows from the identity

$$Z^s - Y^s = (Z - Y) \cdot (Z^{s-1} + Z^{s-2}Y + \dots + ZY^{s-2} + Y^{s-1}) \quad (7)$$

(for $s \geq 1$; for $s = 0$, we get 0). For the induction step: Write $P(\underline{X})$ as a sum of increasing powers of X_1 ,

$$P(X_1, X_2, \dots) = P_0(X_2, \dots) + X_1 P_1(X_2, \dots) + X_1^2 P_2(X_2, \dots).$$

Apply the induction hypothesis to the first term. The remaining terms are of the form $X_1^s Q_s(X_2, \dots)$ with $s \geq 1$; then the difference to be considered is

$$Y_1^s Q_s(Y_2, \dots) - Z_1^s Q_s(Z_2, \dots)$$

which we may write as

$$Y_1^s(Q_s(Y_2, \dots) - Q_s(Z_2, \dots)) + Q_s(Z_2, \dots)(Y_1^s - Z_1^s).$$

The first term in this sum is taken care of by the induction hypothesis, the second term uses the identity (7) which shows that this term is in the ideal generated by $(Z_1 - Y_1)$.

From this follows immediately

Proposition 2.2 *The ideal $J^2 \subseteq B \otimes B$, for $B = k[X_1, \dots, X_n]$ is generated (as a $B \otimes B$ -module) by the elements $(Z_i - Y_i)(Z_j - Y_j)$ (for $i, j = 1, \dots, n$) (identifying $B \otimes B$ with $k[Y_1, \dots, Y_n, Z_1, \dots, Z_n]$).*

(The algebra $(B \otimes B)/J^2$ is the algebra representing the affine scheme “first neighbourhood of the diagonal” for the affine scheme represented by B , alluded to in the introduction.)

Algebra maps $\underline{a} : k[X_1, \dots, X_n] \rightarrow C$ are completely given by an n -tuple of elements $a_i := \underline{a}(X_i) \in C$ ($i = 1, \dots, n$). Let $\underline{b} : k[X_1, \dots, X_n] \rightarrow C$ be similarly given by the n -tuple $b_i \in C$. The decision when $\underline{a} \sim \underline{b}$ can be expressed equationally in terms of these two n -tuples of elements in C , i.e. as a purely equationally described condition on elements $(a_1, \dots, a_n, b_1, \dots, b_n) \in C^{2n}$:

Proposition 2.3 *Consider two algebra maps \underline{a} and $\underline{b} : k[X_1, \dots, X_n] \rightarrow C$. Let $a_i := \underline{a}(X_i)$ and $b_i := \underline{b}(X_i)$. Then we have $\underline{a} \sim \underline{b}$ if and only if*

$$(b_i - a_i) \cdot (b_j - a_j) = 0 \tag{8}$$

for all $i, j = 1, \dots, n$.

Proof. We have that $\underline{a} \sim \underline{b}$ iff the algebra map $\{\underline{a}, \underline{b}\}$ annihilates the ideal J^2 for the algebra $k[X_1, \dots, X_n]$; and this in turn is equivalent to that it annihilates the set of generators for J^2 described in the Proposition 2.2). But $\{\underline{a}, \underline{b}\}((Z_i - Y_i) \cdot (Z_j - Y_j)) = (b_i - a_i) \cdot (b_j - a_j)$, and then the result is immediate.

We therefore also say that the pair of n -tuples of elements in C

$$\begin{bmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{bmatrix}$$

are *neighbours* if (8) holds.

For brevity, we call an n -tuple (c_1, \dots, c_n) of elements in C^n a *vector*, and denote it \underline{c} . Thus a vector (c_1, \dots, c_n) is neighbour of the “zero” vector $\underline{0} = (0, \dots, 0)$ iff $c_i \cdot c_j = 0$ for all i and j .

Remark. Even when $2 \in k$ is invertible, one cannot conclude that $\underline{a} \sim \underline{b}$ follows from $(b_i - a_i)^2 = 0$ for all $i = 1, \dots, n$. For, consider $C := k[\varepsilon_1, \varepsilon_2] = k[\varepsilon] \otimes k[\varepsilon]$ (where $k[\varepsilon]$ is the “ring of dual numbers over k ”, so $\varepsilon^2 = 0$). Then the pair of n -tuples ($n = 2$ here) given by $(a_1, a_2) = (\varepsilon_1, \varepsilon_2)$ and $(b_1, b_2) := (0, 0)$ has $(a_i - b_i)^2 = \varepsilon_i^2 = 0$ for $i = 1, 2$, but $(a_1 - b_1) \cdot (a_2 - b_2) = \varepsilon_1 \cdot \varepsilon_2$, which is not 0 in C .

We already have the notion of when two algebra maps f and $g : B \rightarrow C$ are neighbours, or infinitesimal neighbours. We also say that the pair (f, g) form an *infinitesimal 1-simplex* (with f and g as *vertices*). Also, we have the derived (8) notion of when two vectors in C^n are neighbours, or form an infinitesimal 1-simplex. This terminology is suited for being generalized to defining the notion of *infinitesimal p -simplex* of algebra maps $B \rightarrow C$, or of *infinitesimal p -simplex* of vectors in C^n (for $p = 1, 2, \dots$).

Proposition 2.3 generalizes immediately to infinitesimal p -simplices (where the Proposition is the special case of $p = 1$):

Proposition 2.4 *Consider $p + 1$ algebra maps $\underline{a}_i : k[X_1, \dots, X_n] \rightarrow C$ (for $i = 0, \dots, p$), and let $a_{ij} \in C$ be $\underline{a}_i(X_j)$, for $j = 1, \dots, n$. Then the \underline{a}_i form an infinitesimal p -simplex iff for all $i, i' = 0, \dots, p$ and $j, j' = 1, \dots, n$*

$$(a_{ij} - a_{i'j}) \cdot (a_{ij'} - a_{i'j'}) = 0. \quad (9)$$

3 Affine combinations of mutual neighbours

Let C be a k -algebra. An *affine* combination in a C -module means here a linear combination in the module, with coefficients from C , and where the sum of the coefficients is 1. We consider in particular the C -module $\text{Lin}_k(B, C)$ of k -linear maps $B \rightarrow C$, where B is another k -algebra. Linear combinations of algebra maps are linear, but may fail to preserve the multiplicative structure and 1. However

Theorem 3.1 *Let f_0, \dots, f_p be a $p + 1$ -tuple of mutual neighbour algebra maps $B \rightarrow C$, and let t_0, \dots, t_p be elements of C with $t_0 + \dots + t_p = 1$. Then the affine combination*

$$\sum_{i=0}^p t_i \cdot f_i : B \rightarrow C$$

is an algebra map. Composing with a map $h : C \rightarrow C'$ preserves the affine combination.

Proof. Since the sum is a k -linear map, it suffices to prove that it preserves the multiplicative structure. It clearly preserves 1. To prove that it preserves products $a \cdot b$, we should compare

$$\left(\sum_i t_i f_i(a)\right) \cdot \left(\sum_j t_j f_j(b)\right) = \sum_{i,j} t_i t_j f_i(a) \cdot f_j(b)$$

with $\sum t_i f_i(a \cdot b)$. Now use that $\sum_j t_j = 1$; then $\sum t_i f_i(a \cdot b)$ may be rewritten as

$$\sum_{ij} t_i t_j f_i(a \cdot b).$$

Compare the two displayed double sums: the terms with $i = j$ match since each f_i preserves multiplication. Consider a pair of indices $i \neq j$; the terms with index ij and ji from the first sum contribute $t_i t_j$ times

$$f_i(a) \cdot f_j(b) + f_j(a) \cdot f_i(b), \quad (10)$$

and the terms with index ij and ji from the second sum contribute $t_i t_j$ times

$$f_i(a \cdot b) + f_j(a \cdot b), \quad (11)$$

and the two displayed contributions are equal, since $f_i \sim f_j$ (use the formulation (2)). The last assertion is obvious from the construction.

Theorem 3.2 *Let f_0, \dots, f_p be a $p+1$ -tuple of mutual neighbour algebra maps $B \rightarrow C$. Then any two affine combinations (with coefficients from C) of these maps are neighbours.*

Proof. Let $\sum_i t_i f_i$ and $\sum_j s_j f_j$ be two such affine combinations. To prove that they are neighbours means (using (2)) to prove that for all a and b in B ,

$$\left(\sum_i t_i f_i(a)\right) \cdot \left(\sum_j s_j f_j(b)\right) + \left(\sum_j s_j f_j(a)\right) \cdot \left(\sum_i t_i f_i(b)\right)$$

equals

$$\sum_i t_i f_i(a \cdot b) + \sum_j s_j f_j(a \cdot b).$$

The first of these expressions equals

$$\sum_{ij} t_i s_j f_i(a) \cdot f_j(b) + \sum_{ij} t_i s_j f_j(a) \cdot f_i(b) = \sum_{ij} t_i s_j [f_i(a) \cdot f_j(b) + f_j(a) \cdot f_i(b)]$$

For the second expression, we use $\sum_j s_j = 1$ and $\sum_i t_i = 1$, to rewrite it as the left hand expression in

$$\sum_{ij} t_i s_j f_i(a \cdot b) + \sum_{ij} t_i s_j f_j(a \cdot b) = \sum_{ij} t_i s_j [f_i(a \cdot b) + f_j(a \cdot b)].$$

For each ij , the two square bracket expression match by (2), since $f_i \sim f_j$.

Combining these two results, we have

Theorem 3.3 *Let f_0, \dots, f_p be a $p+1$ -tuple of mutual neighbour algebra maps $B \rightarrow C$. Then in the C -module of C -linear maps $B \rightarrow C$, the affine subspace $\text{Aff}_C(f_0, \dots, f_p)$ of affine combinations (with coefficients from C) of the f_i s consists of algebra maps, and they are mutual neighbours.*

In [1], they describe an ideal $J_{0p}^{(2)}$. It is the sum of ideals J_{rs}^2 in the $p+1$ -fold tensor product $B \otimes \dots \otimes B$, where J_{rs} is the ideal generated by $i_s(b) - i_r(b)$ for $b \in B$ and $r < s$. We shall here denote it just $\bar{J}^{(2)}$ for brevity; it has the property that the $p+1$ inclusions $B \rightarrow B \otimes \dots \otimes B$ become mutual neighbours, when composed with the quotient map $\pi : B \otimes \dots \otimes B \rightarrow (B \otimes \dots \otimes B) / \bar{J}^{(2)}$, and this is in fact the universal $p+1$ tuple of mutual neighbour maps with domain B .

We may, for any given k -algebra B , encode the construction of Theorem 3.1 into one single canonical map which does not mention any individual $B \rightarrow C$, by using the universal $p+1$ -tuple, and the generic $p+1$ tuple of coefficients with sum 1, meaning $(X_0, X_1, \dots, X_p) \in k[X_1, \dots, X_p]$ (where X_0 denotes $1 - (X_1 + \dots + X_p)$); namely as a k -algebra map

$$B \rightarrow (B^{\otimes p+1} / \bar{J}^{(2)}) \otimes k[X_1, \dots, X_p]. \quad (12)$$

For, by the Yoneda Lemma, this is equivalent to giving a (set theoretical) map, natural in C ,

$$\text{hom}((B^{\otimes p+1} / \bar{J}^{(2)}) \otimes k[X_1, \dots, X_p], C) \rightarrow \text{hom}(B, C),$$

(where hom denotes the set of k -algebra maps). An element on the left hand side is given by a $p+1$ -tuple of mutual neighbouring algebra maps $f_i : B \rightarrow C$, together

with a p -tuple (t_1, \dots, t_p) of elements in C . With $t_0 := 1 - \sum_1^p t_i$, such data produce an element $\sum_0^p t_i \cdot f_i$ in $\text{hom}(B, C)$, by Theorem 3.1, and the construction is natural in C by the last assertion in the Theorem.

The affine scheme defined by the algebra $B^{\otimes p+1}/\bar{J}^{(2)}$ is (essentially) called $\Delta_B^{(p)}$ in [1], and, (in axiomatic context, and for manifolds, in a suitable sense), the corresponding object is called $M_{[p]}$ in [4] and $M_{(1,1,\dots,1)}$ in [3] I. 18 (for suitable M).

4 Affine combinations in a k -algebra C

The constructions and results of the previous Section concerning infinitesimal p -simplices of algebra maps $B \rightarrow C$, specializes (by taking $B = k[X_1, \dots, X_n]$, as in Section 2) to infinitesimal p -simplices of vectors in C^n ; such a p -simplex is conveniently exhibited in a $(p+1) \times n$ matrix with entries a_{ij} from C :

$$\begin{bmatrix} a_{01} & \dots & a_{0n} \\ a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{pn} \end{bmatrix}.$$

We may of course form affine (or even linear) combinations, with coefficients from C , of the rows of this matrix, whether or not the rows are mutual neighbours. But the same affine combination of the corresponding algebra maps is in general only a k -linear map, not an algebra map. However, if the rows are mutual neighbours in C^n , and hence the corresponding algebra maps are mutual neighbouring algebra maps $k[X_1, \dots, X_n] \rightarrow C$, we have, by Theorem 3.1 that the affine combinations of the rows of the matrix corresponds to the similar affine combination of the algebra maps. For, it suffices to check their equality on the X_i s, since the X_i s generate $k[X_1, \dots, X_n]$ as an algebra. Therefore, the Theorems 3.2 and 3.3 immediately translate into theorems about $p+1$ -tuples of mutual neighbouring n -tuples of elements in the algebra C ; recall that such a $p+1$ -tuple may be identified with the rows of a $(p+1) \times n$ matrix with entries from C , satisfying the equations (9).

Theorem 4.1 *Let the rows of a $(p+1) \times n$ matrix with entries from C be mutual neighbours. Then any two affine combinations (with coefficients from C) of these rows are neighbours. The set of all such affine combinations form an affine subspace of the C -module C^n .*

Let us consider in particular the case where the 0th row of a $(p+1) \times n$ matrix is the zero vector $(0, \dots, 0)$. Then the following is an elementary calculation:

Proposition 4.2 *Consider a $(p+1) \times n$ matrix $\{a_{ij}\}$ as above, but with $a_{0j} = 0$ for $j = 1, \dots, n$. Then the rows form an infinitesimal p -simplex iff the conjunction of*

$$a_{ij} \cdot a_{i'j'} + a_{i'j} \cdot a_{ij'} = 0 \text{ for all } i, i' = 1, \dots, p, j, j' = 1, \dots, n. \quad (13)$$

hold, and

$$a_{ij} \cdot a_{ij'} = 0 \text{ for all } i = 1, \dots, p, j = 1, \dots, n \quad (14)$$

If 2 is invertible in C , the equations (14) follow from (13).

Proof. The last assertion follows by putting $i = i'$ in (13), and dividing by 2. Assume that the rows of the matrix form an infinitesimal p -simplex. Then (14) follows from $\underline{a}_i \sim \underline{0}$. The equation which asserts that $\underline{a}_i \sim \underline{a}_{i'}$ (for $i, i' = 1, \dots, p$) is

$$(a_{ij} - a_{i'j}) \cdot (a_{ij'} - a_{i'j'}) = 0 \text{ for all } j, j' = 1, \dots, n.$$

Multiplying out gives four terms, two of which vanish by virtue of (14), and the two remaining add up to (minus) the sum on the left of (13). For the converse implication, (14) give that the last p rows are $\sim \underline{0}$; and (14) and (13) jointly give that $\underline{a}_i \sim \underline{a}_{i'}$, by essentially the same calculation which we have already made.

When $\underline{0}$ is one of the vectors in a $p+1$ -tuple, any linear combination of the remaining p vectors has the same value as a certain affine combination of all $p+1$ vectors, since the coefficient for $\underline{0}$ may be chosen arbitrarily without changing the value of the linear combination. Therefore the results on affine combinations of the rows in the $(p+1) \times n$ matrix with $\underline{0}$ as top row immediately translate to results about linear combinations of the remaining rows, i.e. they translate into results about $p \times n$ matrices, satisfying the equations (13) and (14); even the equations (13) suffice, if 2 is invertible. In this form, the results were obtained in the preprint [6], and are stated here for completeness. We assume that $2 \in k$ is invertible.

We use the notation from [3] I.16 and I. 18, where set of $p \times n$ matrices $\{a_{ij}\}$ satisfying (13) was denoted $\tilde{D}(p, n) \subseteq C^{p \times n}$ (we there consider algebras C over $k = \mathbb{Q}$, so (14) follows). In particular $\tilde{D}(2, 2)$ consists of matrices of the form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{with} \quad a_{11} \cdot a_{22} + a_{12} \cdot a_{21} = 0.$$

Note that the determinant of such a matrix is 2 times the product of the diagonal entries. And also note that $\tilde{D}(2, 2)$ is stable under transposition of matrices.

The notation $\tilde{D}(p, n)$ may be consistently augmented to the case where $p = 1$; we say $(a_1, \dots, a_n) \in \tilde{D}(1, n)$ if it is neighbour of $\underline{0} \in C^n$, i.e. if $a_j \cdot a_{j'} = 0$ for all $j, j' = 1, \dots, n$. (In [3], $\tilde{D}(1, n)$ is also denoted $D(n)$, and $D(1)$ is denoted D .)

It is clear that a $p \times n$ matrix belongs to $\tilde{D}(p, n)$ precisely when all its 2×2 sub-matrices do; this is just a reflection of the fact that the defining equations (13) only involve two row indices and two column indices at a time. From the transposition stability of $\tilde{D}(2, 2)$ therefore follows that transposition $p \times n$ matrices takes $\tilde{D}(p, n)$ into $\tilde{D}(n, p)$.

Note that each of the rows of a matrix in $\tilde{D}(p, n)$ is a neighbour of $\underline{0} \in C^n$.

The results about affine combinations now get the following formulations in terms of linear combinations of the rows of matrices in $\tilde{D}(p, n)$:

Theorem 4.3 *Given a matrix $X \in \tilde{D}(p, n)$. Let a $(p+1) \times n$ matrix X' be obtained by adjoining to X a row which is a linear combination of the rows of X . Then X' is in $\tilde{D}(p+1, n)$.*

5 Geometric meaning

This Section contains essentially only reformulations of the previous material into geometric terms, and thereby it also contains some motivation of the notions.

In any category \mathcal{E} , a map $f : I \rightarrow M$ may be thought of either as an I -parametrized family of elements (or points) of M , or as an M -valued function on I . This terminology is convenient in particular when \mathcal{E} is in some sense a category of spaces, and is even traditional in algebraic geometry, for instance when \mathcal{E} is the *dual* of the category \mathcal{A} of commutative k -algebras. The category \mathcal{E} is in this case essentially the category of *affine schemes* over k . We elaborate a little on this terminology for this specific case. When a commutative k -algebra B is seen in the dual category \mathcal{E} , it is often denoted $\text{spec } B$ or \overline{B} . If X is an object in \mathcal{E} , the corresponding algebra is often denoted $O(X)$, and called the *algebra of functions* on X ; more precisely, the algebra of *scalar* valued functions on X .

The category \mathcal{E} is in this case a category equipped with a canonical commutative ring object R , namely $\overline{k[X]}$, whose geometric meaning is that it is the number line. The ring structure of R in \mathcal{E} comes about from the canonical co-ring structure of $k[X]$ in the category of commutative k -algebras. Now ring structure on the geometric line is elementary and well understood, since the time of Euclid, essentially, whereas the notion of coring is not elementary, and is a much more recent invention. This is why (\mathcal{E}, R) is well suited to axiomatic abstraction, as in [3].

The reason why $\text{hom}_{\mathcal{E}}(I, R)$ is a commutative ring (even a k -algebra) in \mathcal{E} is that it is isomorphic to $O(I)$; for,

$$\text{hom}_{\mathcal{E}}(I, R) \cong \text{hom}_{\mathcal{E}}(I, \overline{k[X]}) \cong \text{hom}_{\mathcal{A}}(k[X], O(I)) \cong O(I), \quad (15)$$

the last isomorphism because $k[X]$ is the free k -algebra in one generator X .

Having a commutative ring object like R in a category \mathcal{E} is the first necessary condition for having the wonderful tool of coordinates available for the geometry in \mathcal{E} .

Thus we have the *coordinate vector spaces* R^n ; in the category of affine schemes, this is $\overline{k[X_1, \dots, X_n]}$. This object is an R -module object. The R -module structure may be described in the same way, in terms of \overline{C} -parametrized points of R^n , as when we described the ring structure of R in such terms; a \overline{C} -parametrized point of R^n amounts to an n -tuple of elements in the algebra C , and C -parametrized points of R amount to elements of C . So the R -module structure of R^n comes about from the C -module structure of C^n , for arbitrary C .

If C is a finitely presented k -algebra, the object (space) \overline{C} embeds into some R^n , since a finite presentation of C , with n generators gives rise to a surjective (hence epimorphic) algebra map $k[X_1, \dots, X_n] \rightarrow C$.

Since we have the notion of “neighbours” for algebra maps, we have a notion of neighbours for maps in \mathcal{E} ; and it is preserved by pre- and post-composition. In particular, the embedding map $e : \overline{C} \rightarrow R^n$, obtained from a presentation of C with n generators, preserves the property of being neighbours, for parametrized families of points of \overline{C} . The embedding also *reflects* the neighbour relation, in the sense that if x and y are points of \overline{C} , and $e(x) \sim e(y)$, then $x \sim y$. This is just a reformulation of (6).

Note the traditional replacement of the notation $e \circ x$ by $e(x)$. This is the “symbolic” counterpart of considering maps $I \rightarrow M$ as (I -parametrized) *points*² of M . Furthermore, it is tradition in algebraic geometry not always to be specific about the space I of parameters for a parametrized point $I \rightarrow M$; thus, one talks about “points (x, y) of the unit circle S given by $x^2 + y^2 = 1$ ”, without explicit mention of whether it one means a real point, a rational point, a complex point, \dots ; therefore, I is omitted from notation, and one writes $(x, y) \in S$. This in particular applies when the statement or notion applies to *any* (parametrized) point of M , regardless of its parameter space I .

²In algebraic geometry, the terminology “ I -valued point of M ” is also used, see e.g. [1] p. 209. In [3], Part II, such a thing is called a “generalized element of M , defined at stage I ”, and a more elaborate description of the ‘logic’ of generalized elements is presented.

An example of this usage is for the neighbour relation in affine schemes. Consider two maps f and g between k -algebras B and C , as in (1). In the category of affine schemes, these are then neighbour maps \overline{f} and \overline{g} : $\overline{C} \rightarrow \overline{B}$, i.e. neighbour points of \overline{B} (parametrized by \overline{C}); we write $\overline{f} \in \overline{B}$, $\overline{g} \in \overline{B}$.

With this usage, Proposition 2.3 may be reformulated as

Proposition 5.1 *Given two points (a_1, \dots, a_n) and $(b_1, \dots, b_n) \in R^n$. Then they are neighbours iff*

$$(b_i - a_i) \cdot (b_j - a_j) = 0 \quad (16)$$

for all $i, j = 1, \dots, n$.

Here the (common) parameter space \overline{C} of the a_i s and b_i s is not mentioned explicitly; it could be any affine scheme. Note that (16) is typographically the same as (8); in (16), a_i and b_j are (parametrized) points of R (parametrized by \overline{C}), in (8), they are elements in the algebra C ; but these data correspond, by (15), and this correspondence preserves algebraic structure.

Similarly, Proposition 2.4 gets the reformulation:

Proposition 5.2 *A $p+1$ -tuple $\{a_{ij}\}$ of points in R^n form an infinitesimal p -simplex iff the equations (9) hold.*

This formulation, as the other formulations in “synthetic” terms, are the ones that are suited to axiomatic treatment, as in Synthetic Differential Geometry, which almost exclusively³ assumes a given commutative ring object R in a category \mathcal{C} , preferably a topos, as a basic ingredient in the axiomatics. (The category \mathcal{C} of affine schemes is not a topos, but the category of presheaves on \mathcal{C} is, and it, and some of its subtoposes, are the basic categories considered in modern algebraic geometry, like [2].)

As a further illustration of the “synthetic” language, the algebraic formulation of the neighbour relation between algebra maps given in (3) (assuming $2 \in k$ is invertible) may be rendered:

Proposition 5.3 *For any affine scheme \overline{B} , scalar valued functions on \overline{B} detect when points x and y of \overline{B} are neighbours; i.e. if $\alpha(x) \sim \alpha(y)$ for all $\alpha : \overline{B} \rightarrow R$, then $x \sim y$.*

³Exceptions are found in [9] (where R is constructed out of an assumed infinitesimal object T); and in [5] and [7], where part of the reasoning does not assume any algebraic notions.

Here x and y are points of \overline{B} , say parametrized by \overline{C} , i.e. they are maps $\overline{C} \rightarrow \overline{B}$, so they correspond to algebra maps f and $g : B \rightarrow C$; and $\alpha : \overline{B} \rightarrow R$ corresponds to $a \in B$.

The fact (6) gets the following formulation:

Proposition 5.4 *Given an affine scheme \overline{B} . Then for any finite presentation (with n generators, say) of the algebra, the corresponding embedding $e : \overline{B} \rightarrow R^n$ (preserves and) reflects the relation \sim .*

So for x and y points in R^n , they come via e from a pair of neighbour points in \overline{B} iff they satisfy 1) the equations in n variables defining B in the presentation; and 2) satisfy the equations for being neighbours in R^n . In other words, the intrinsically defined neighbour relation on \overline{B} (essentially (1)) may be described purely equationally, using an finite equational presentation of B .

Or, in more elementary terms, which the synthetic tradition is very apt for utilizing: Given a finite set of equations with coefficients from R . If of x_0, \dots, x_p are points in R^n and each of them satisfies the equations, then so does any affine combination of them, provided the points are mutual neighbours. But note that Proposition 5.4, together with the constructions of Sections 1 and 3 allow us to conclude that the neighbour conditions, and the point constructed by affine combinations in R^n , is intrinsic to the affine scheme \overline{B} in question, and does not depend on an equational presentation of B .

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